Lezione 16: Limiti notevoli - Esercizi

INDICE

LIMITI NOTEVOLI:

I° GRUPPO

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1 \qquad \lim_{x\to 0}\frac{(1+x)^\alpha-1}{x} = \alpha$$

II° GRUPPO

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\tan x}{x} = 1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0}\frac{\tan x - \sin x}{x^3} = \frac{1}{2} \qquad \lim_{x\to 0}\frac{\arctan x}{x} = 1 \qquad \lim_{x\to 0}\frac{\arcsin x}{x} = 1$$

ESERCIZI

A) $\lim_{x\to+\infty}\sin x$

B) $\lim_{x\to 0}\frac{1}{x}$

C) $\lim_{x\to 0^+}\frac{1}{x}$

$\lim_{x\to+\infty}\frac{\tan(\arctan x)}{x}$

D) $\lim_{x\to+\infty}\frac{\sin x}{x}$ $\qquad -\frac{1}{x} < \frac{\sin x}{x} < +\frac{1}{x}$, $\quad \downarrow 0 \quad \downarrow 0 \quad \downarrow 0$

E) $\lim_{x\to+\infty}\frac{\arctan(\tan x)}{x}$

F) $\lim_{x\to+\infty}\left(1+\frac{\sin x}{x}\right)^x$

G) $\lim_{x\to 0}\frac{\sin 3x}{\sqrt{1+x}-1}$

H) $\lim_{x\to 0}\frac{\cos(\sin x)-1}{(e^{3x}-1)\ln(1+4x)}$

I) $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\cos x}{e^{x^2}-e^x}$

J) $\lim_{x\to 1}\frac{4^x-2^{x+1}}{\ln x}$

K) $\lim_{x\to+\infty}|\sin(\sin x)|^x$

L) $\lim_{x\to+\infty}|\cos(\cos x)|^x$

M) $\lim_{x\to-\frac{\pi}{2}}\frac{\sin x+1}{\cos 4x-1}$

N) $\lim_{x\to+\infty}\left(\cos\frac{1}{x}\right)^{x^2}$

O) $\lim_{x\to 0^+}\left(\frac{\pi}{2}+\arctan(\ln x)\right)\cdot\ln\frac{1}{x}$

P) $\lim_{x\to 0^-}\frac{\tan x-\sin(x+x^2)}{e^{x^2}-\cos x+e^{\frac{1}{x}}}$

Q) $\lim_{x\to+\infty}\left|\sin x\cdot\cos\sqrt{x^2+1}\right|^x$

T. DATO $A \subset \mathbb{R}$ x_0 t.c. $\forall \delta > 0$ SIA $(x_0, x_0+\delta)$ CHE $(x_0-\delta, x_0)$ INTERSECANO A. ED $f: A \to \mathbb{R}$. ALLORA È EQUIV. DIRE CHE

1) $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}^*$

2) $\left[\lim_{x \to x_0^+} f(x) = \ell\right]$ E $\boxed{\lim_{x \to x_0^-} f(x) = \ell}$

DIM

(1) $\iff$ $\forall I$ INTORNO DI ℓ $\exists \delta > 0$ t.c. $\left.\begin{array}{l} x \in (x_0-\delta, x_0+\delta) \\ x \neq x_0 \\ x \in A \end{array}\right\} \Rightarrow f(x) \in I$

(2) $\iff$ $\forall I$ INTORNO DI ℓ $\exists \delta_1 > 0$ t.c. $\left.\begin{array}{l} x \in (x_0-\delta_1, x_0) \\ x \in A \end{array}\right\} \Rightarrow f(x) \in I$

$\exists \delta_2 > 0$ t.c. $\left.\begin{array}{l} x \in (x_0, x_0+\delta_2) \\ x \in A \end{array}\right\} \Rightarrow f(x) \in I$

$\delta' = \min(\delta_1, \delta_2)$

$x \in A$, $x \in (x_0-\delta', x_0+\delta')$, $x \neq x_0$

(2) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2).

LIM. NOTEVOLI

1) $\lim_{x \to +\infty} \left(1+\frac{1}{x}\right)^x = e$

$\boxed{-\infty}$

$a_n \to +\infty \quad \left(1+\frac{1}{a_n}\right)^{a_n} \to e$

$\boxed{a_n \to -\infty}$

$\frac{1}{f(x)} \to +\infty$

$f(x) \to 0^+$

2) $\boxed{\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e}$

$\boxed{x \to 0^+ \Rightarrow \frac{1}{x} \to +\infty}$

$\lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = \lim_{x \to 0^+} \left(1+\frac{1}{\frac{1}{x}}\right)^{\frac{1}{x}} \overset{y=\frac{1}{x}}{=} \lim_{y \to +\infty} \left(1+\frac{1}{y}\right)^y = e$

$\lim_{x \to 0^-} (1+x)^{\frac{1}{x}} = \lim_{x \to 0^-} \left(1+\frac{1}{\frac{1}{x}}\right)^{\frac{1}{x}} \overset{y=\frac{1}{x}}{=} \lim_{y \to -\infty} \left(1+\frac{1}{y}\right)^y = e$

3) $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$

$y = (1+x)^{\frac{1}{x}}$

$\lim_{x \to 0} \frac{1}{x} \cdot \ln(1+x) = \lim_{x \to 0} \boxed{\ln}\left((1+x)^{\frac{1}{x}}\right) = \lim_{y \to e} \ln(y) = 1$

4) $\lim_{x \to 0} \frac{\boxed{e^x - 1}}{x} = \lim_{x \to 0} \frac{1}{\frac{\ln(1+(e^x-1))}{(e^x-1)}} \overset{y=e^x-1}{=} \lim_{y \to 0} \frac{1}{\boxed{\frac{\ln(1+y)}{y}}} = \frac{1}{1}$ $(=1$

$\ln(e^x) = \ln(e^x - 1 + 1) = \ln(1+(e^x-1))$.

5) $\lim_{x\to0} \frac{(1+x)^\alpha - 1}{x} = \alpha$

$$= \lim_{x\to0} \frac{e^{\ln(1+x)^\alpha}-1}{x} =$$

$$= \lim_{x\to0}\frac{e^{\alpha\ln(1+x)}-1}{x} =$$

$$= \lim_{x\to0} \underbrace{\frac{e^{\alpha\ln(1+x)}-1}{\alpha\ln(1+x)}}_{1}\cdot \underbrace{\frac{\ln(1+x)}{x}}_{1}\cdot \underbrace{\alpha}_{\alpha} = \alpha$$

$$\lim_{x\to0}\frac{e^{\alpha\ln(1+x)}-1}{\alpha\ln(1+x)} = \lim_{y\to0}\frac{e^y-1}{y} = 1$$

$y=\alpha\ln(1+x) \xrightarrow[x\to0]{} 0$

$$\lim_{x\to0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to0}\frac{\cancel{1}+x\cancel{-1}}{x(\sqrt{1+x}+1)} = \lim_{x\to0}\frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$

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6) $\boxed{\lim_{x\to0}\frac{\sin x}{x} = 1}$

$$\lim_{x\to0^+}\frac{\sin x}{x} = \lim_{x\to0^+}\frac{1}{\frac{x}{\sin x}} = \frac{1}{1} = 1$$

$y=-x$

$$\lim_{x\to0^-}\frac{\sin x}{x} = \lim_{x\to0^-}\frac{\sin(-x)}{-x} = \lim_{y\to0^+}\frac{\sin y}{y} = 1$$

$x\in\left(0,\frac{\pi}{2}\right)$

$$\boxed{\sin x < x < \tan x}$$

2) $\lim_{x\to0}\frac{\tan x}{x}=\lim_{x\to0}\underbrace{\frac{\sin x}{x}}_{\downarrow\,1}\cdot\underbrace{\frac{1}{\cos x}}_{\downarrow\,1}=1\cdot1=1$

3) $\lim_{x\to0}\frac{1-\cos x}{x^2}=\lim_{x\to0}\frac{(1-\cos x)(1+\cos x)}{x^2\cdot(1+\cos x)}=\lim_{x\to0}\frac{\sin^2 x}{x^2\cdot(1+\cos x)}=$

$$=\lim_{x\to0}\Bigg(\underbrace{\left(\frac{\sin x}{x}\right)^2}_{\downarrow\,1^2}\cdot\underbrace{\frac{1}{1+\cos x}}_{\downarrow\,\frac{1}{2}}\Bigg)=1^2\cdot\frac{1}{2}=\frac{1}{2}$$

4) $\lim_{x\to0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to0}\frac{\sin x\left(\frac{1}{\cos x}-1\right)}{x^3}=$

$$=\lim_{x\to0}\frac{\sin x\,\frac{1-\cos x}{\cos x}}{x^3}=$$

$$=\lim_{x\to0}\Bigg(\underbrace{\frac{\sin x}{x}}_{\downarrow\,1}\cdot\underbrace{\frac{1-\cos x}{x^2}}_{\downarrow\,\frac{1}{2}}\cdot\underbrace{\frac{1}{\cos x}}_{\downarrow\,1}\Bigg)=\frac{1}{2}$$

5) $\lim_{x\to0}\frac{\arctan x}{x}=\lim_{x\to0}\frac{\arctan x}{\tan(\arctan x)}\overset{y=\arctan x}{=}\lim_{y\to0}\frac{y}{\tan(y)}=$

$$= \lim_{y \to 0} \frac{1}{\frac{\sin y}{y}} = \frac{1}{1} = 1$$

$y = \arcsin x$

$$\text{b)} \lim_{x \to 0} \frac{\arcsin x}{x} = \lim_{x \to 0} \frac{\arcsin x}{\sin(\arcsin x)} \overset{\Downarrow}{=} \lim_{y \to 0} \frac{y}{\sin y} = \lim_{y \to 0} \frac{1}{\frac{\sin y}{y}} = 1$$

$\boxed{\text{EX } \widetilde{\text{II}}}$

$$\lim_{n \to +\infty} \sin n$$

$$a_n = 2\pi n \to +\infty$$

$$b_n = 2\pi n + \frac{\pi}{2} \to +\infty$$

$$\sin(a_n) = 0 \to 0$$

$$\sin(b_n) = 1 \to 1$$

$\boxed{B}$ $\lim_{x \to 0} \frac{1}{x}$

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

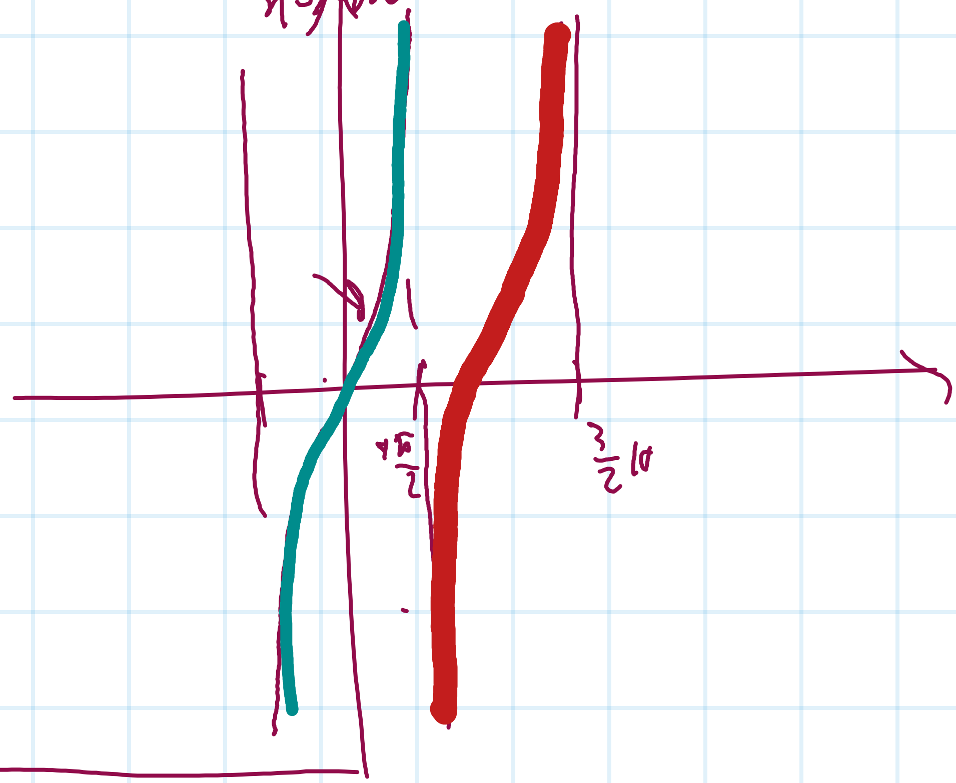
$\neq$

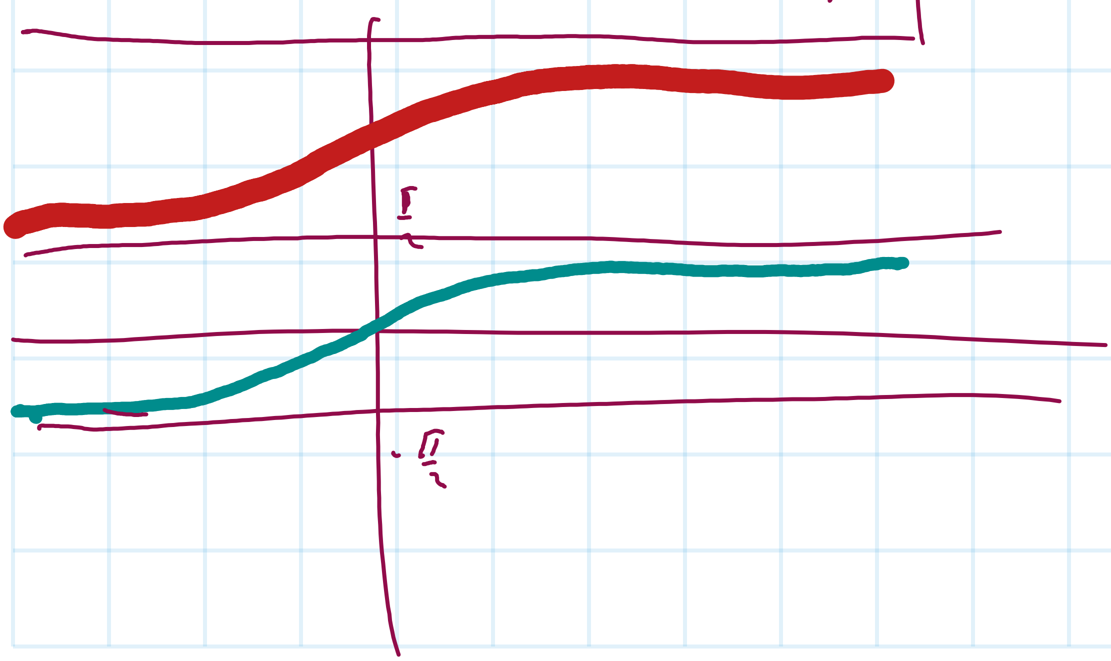
$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

1) $\lim_{x \to +\infty} \frac{\arctan(\tan x)}{x} =$

$$\underbrace{\frac{-\frac{\pi}{2}}{x}}_{\to 0} < \frac{\arctan(\text{[illegible]})}{x} < \underbrace{\frac{\frac{\pi}{2}}{x}}_{\to 0}$$

2) $\lim_{x \to +\infty} \frac{\tan(\arctan x)}{x} = \lim_{x \to +\infty} \frac{x}{x} = 1$





$\boxed{H}$ $\lim\limits_{x\to 0} \dfrac{\cos(\sin x)-1}{(e^{3x}-1)\ln(1+4x)}$

$$= \lim_{x\to 0} \underbrace{\frac{\cos(\sin x)-1}{(\sin x)^2}}_{-\frac{1}{2}} \cdot \underbrace{\frac{3x}{e^{3x}-1}}_{1} \cdot \underbrace{\frac{4x}{\ln(1+4x)}}_{1} \cdot \underbrace{\frac{(\sin x)^2}{12x^2}}_{\frac{1}{12}}$$

$\frac{3x}{e^{3x}-1} \to 1$

$\frac{4x}{\ln(1+4x)} \to 1$

$\frac{(\sin x)^2}{12x^2} \to \frac{1}{12} = \frac{1}{12}\left(\frac{\sin x}{x}\right)^2$

$$-\frac{1-\cos(\sin x)}{(\sin x)^2}$$

$y = \sin x$

$$\lim_{x\to 0} \boxed{} = \lim_{y\to 0} -\frac{1-\cos y}{y^2} = -\frac{1}{2}$$

$\frac{1-\cos y}{y^2} \to \frac{1}{2}$

$$= -\frac{1}{24}$$

$\boxed{I}$ $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^2} - \cos x}{e^{x^2} - e^x}$

$$= \lim_{x \to 0} \frac{1 + x^2 - \cos^2 x}{\left(e^{x^2 - x} - 1\right) e^x \cdot \left(\sqrt{1+x^2} + \cos x\right)} \approx$$

$$\lim_{x \to 0} \left(\frac{\overbrace{(1 - \cos^2 x)}^{\sin^2 x} + x^2}{e^{(x^2 - x)} - 1} \cdot \frac{1}{e^x \cdot \left(\sqrt{1+x^2} + \cos x\right)} \right) \to \frac{1}{2}$$

$$\frac{\sin^2 x}{x^2} \cdot \frac{x^2}{e^{x^2 - x} - 1} + \frac{x^2}{e^{(x^2 - x)} - 1}$$

$\downarrow$

1

$$\frac{x^2 - x}{e^{x^2 - x} - 1} \cdot \frac{x^2}{x^2 - x}$$

$\to 1$

$$f = o(g)$$

$$\lim_{x \to x_0} \frac{f(x) + g(x)}{h(x)} = \lim_{x \to x_0} \frac{\boxed{g(x)}\left(\frac{f(x)}{g(x)} + 1\right)}{h(x)}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - \cos^2 x}{\boxed{e^x - e^{x^2}}} = e^{x^2}\left(e^{x - x^2} - 1\right)$$

$$= \lim_{x \to 0} \frac{\text{-- -- -- ——}}{\underbrace{\left(e^x - 1\right)}\,\cancel{\left(1 - e^{x \cdot x}\right)}}$$

$$\boxed{\frac{e^{x^2} - 1}{x^2}} \cdot x^2 \qquad \boxed{\frac{e^x - 1}{x} \cdot x}$$