Analisi Matematica 1 - 23 Ottobre 2024 - docente: Callegari

Lezione 14: Limite di funzioni in una variabile

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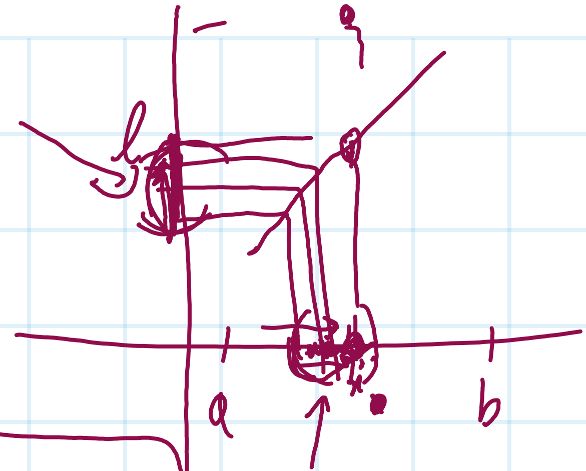
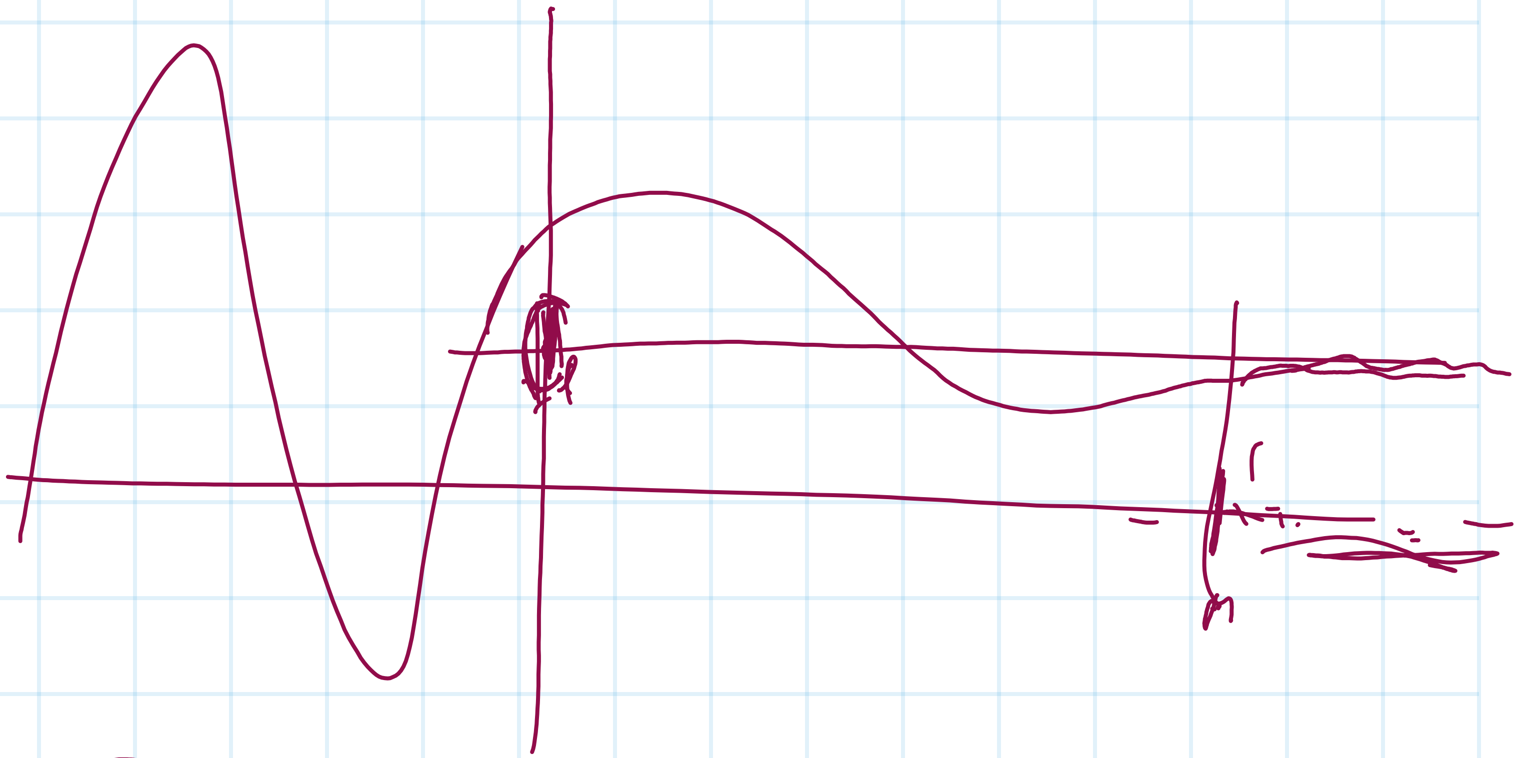
DEF 1

DATA $f: A \to \mathbb{R}$ (con $A \subseteq \mathbb{R}$) E x_0 DI ACC. PER A.

DIREMO CHE "$\lim_{x \to x_0} f(x) = l$" $\in \mathbb{R}$



$$\boxed{\forall \varepsilon > 0 \ \exists \delta > 0 \text{ t.c. } \left.\begin{array}{l} |x - x_0| < \delta \\ x \neq x_0 \\ x \in A \end{array}\right\} \Rightarrow |f(x) - l| < \varepsilon}$$

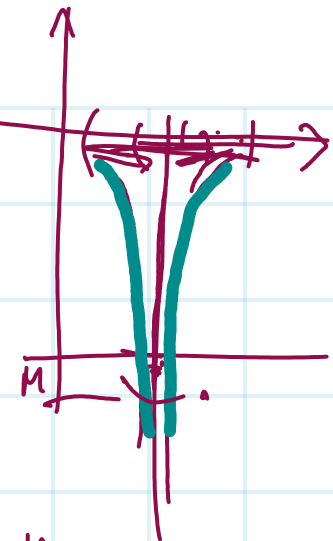


DEF DATO $A \subseteq \mathbb{R}$ NON SUPERIORMENTE LIMITATO, E $f: A \to \mathbb{R}$

$l \in \mathbb{R}$, DIREMO CHE $\boxed{\text{"}\lim_{x \to +\infty} f(x) = l\text{"}}$

$$\forall \varepsilon > 0 \ \exists M > 0 \ (\in \mathbb{R}) \text{ t.c. } \left.\begin{array}{l} x > M \\ x \in A \end{array}\right\} \Rightarrow |f(x) - l| < \varepsilon$$

DEF. DATI $A \subset \mathbb{R}$, x_0 DI ACC. PER A. E $f: A \to \mathbb{R}$



DIREMO CHE $\boxed{\lim_{x \to x_0} f(x) = -\infty}$ SE.

$$\boxed{\forall M < 0 \ (\in \mathbb{R}) \quad \exists \delta > 0 \quad t.c. \quad \left.\begin{matrix} |x - x_0| < \delta \\ x \in A \\ x \neq x_0 \end{matrix}\right\} \Rightarrow f(x) < M}$$

$(0, +\infty)$

$I_{+\infty}$ = SEMIRETTA DESTRA

$I_{-\infty}$ = SEM. SINISTRA

$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$

$+\infty$ È DI ACC. PER $A \subset \mathbb{R}$ SE $\forall M > 0$

$$A \cap (M, +\infty) \neq \emptyset$$

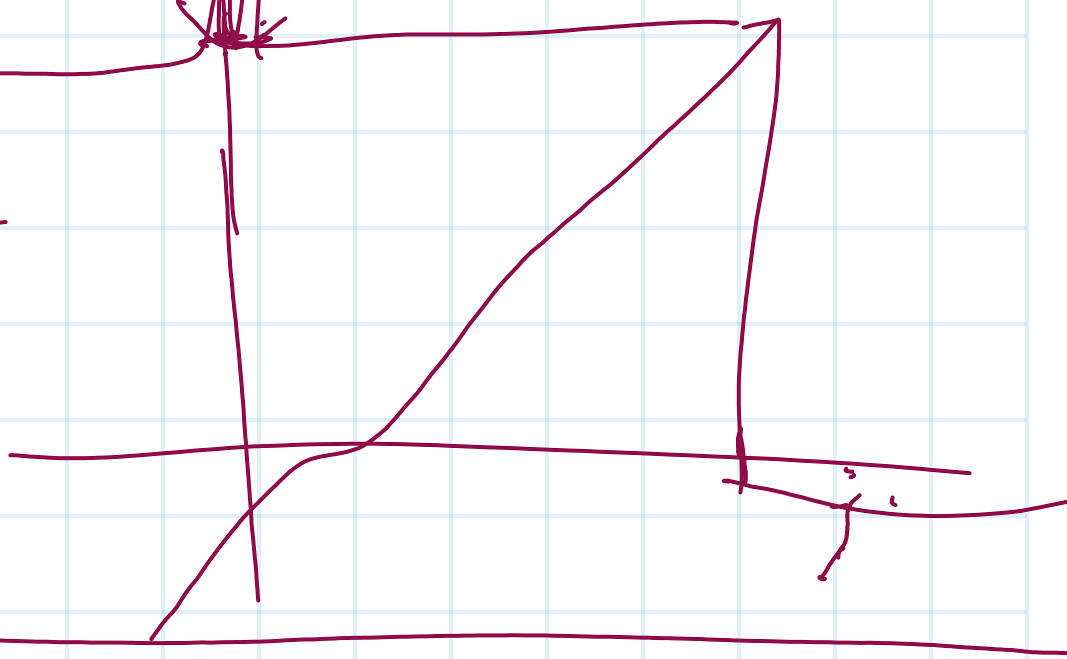
$\boxed{\text{DEF.}}$ DATE $A \subset \mathbb{R}$, $x_0 \in \mathbb{R}^*$ (DI ACC. PER A), $\ell \in \mathbb{R}^*$, $f : A \to \mathbb{R}$;

DIREMO CHE "$\lim_{x \to x_0} f(x) = \ell$" SE:

$\forall I$ INTORNO DI ℓ $\exists J$ INTORNO DI x_0 t.c.

$$f(J \cap (A - \{x_0\})) \subseteq I.$$

"$\lim_{x \to +\infty} f(x) = +\infty$"



$\boxed{\text{T.}}$ (PONTE)

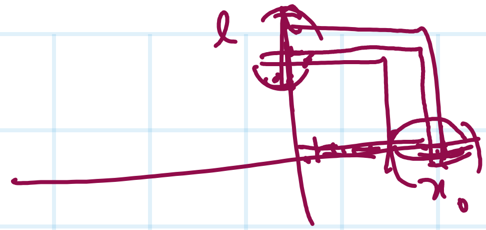
DATI $A \subset \mathbb{R}$ x_0 DI ACC. PER A, $f : A \to \mathbb{R}$ $\ell \in \mathbb{R}$

È EQUIVALENTE AFFERMARE CHE

1) "$\lim_{x \to x_0} f(x) = \ell$"

2) $\forall (a_n)$ A VALORI IN $A - \{x_0\}$, $(a_n \to x_0) \Rightarrow f(a_n) \to \ell$

$\boxed{\text{DIM}}$ $(1) \Rightarrow (2)$

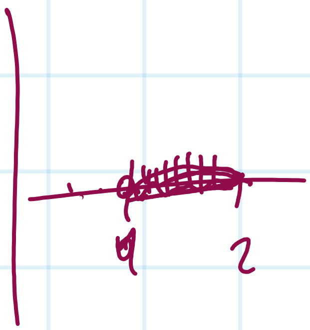


$\rightarrow$ $\forall I$ intorno di l $\exists J$ INTORNO DI x_0 t.c. $\boxed{f(J \cap A - \{x_0\}) \subset I}$

$\forall (a_n)$ IN $A - \{x_0\}$ DIRE CHE $a_n \to x_0$

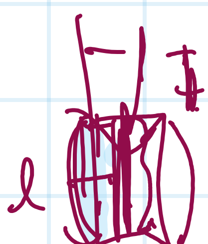
SIGNIFICA DIRE CHE $\overset{\text{DEF. lim}}{\checkmark}$ $a_n \in (J \cap (A - \{x_0\}))$

DEF lim n $f(a_n) \in I$



$f(J_n \cap (A - \{x_0\})) \not\subset I$

$(2) \Rightarrow (1)$



$\rightarrow \boxed{J_n} = (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$

quindi $a_n \in J_n \cap \ldots$

$f(a_n) \notin I$

$\neg\left(\forall I \text{ int. di } l \ \exists J \text{ int. di } x_0 \text{ t.c. } f(J \cap (A - \{x_0\})) \subset I\right)$

$\exists I$ int. di l t.c. $\forall J$ int. di x_0 $\exists x \in J \cap (A - \{x_0\})$ t.c.

$f(x) \notin I$

$(2) \Rightarrow (1)$ P.A. $\exists I$ INT. DI l t.c. $\forall J$ INT. DI x_0 $f(J \cap (A - \{x_0\})) \not\subset I$

IN PARTICOLARE $\forall n \in \mathbb{N}$ $f(J_n \cap (A - \{x_0\})) \not\subset I$

$\left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right)$

QUINDI $\exists a_n \in \underbrace{J_n \cap (A - x_0)}$ t.c. $\underbrace{f(a_n) \notin I}$

$a_n \to x_0$ (?) (?)

$\forall \varepsilon > 0$ def. $|a_n - x_0| < \varepsilon$?

$|a_n - x_0| < \frac{1}{n}$

$f(a_n) \not\to l$

HO TROVATO (a_n) t.c. $\underbrace{a_n \to x_0 \text{ MA } f(a_n) \not\to l}$

PER $x \to x_0$ $f(x) \to l$ (S)

$g(x) \to L$

$\Rightarrow$

$f(x) \cdot g(x) \to l \cdot L$

?

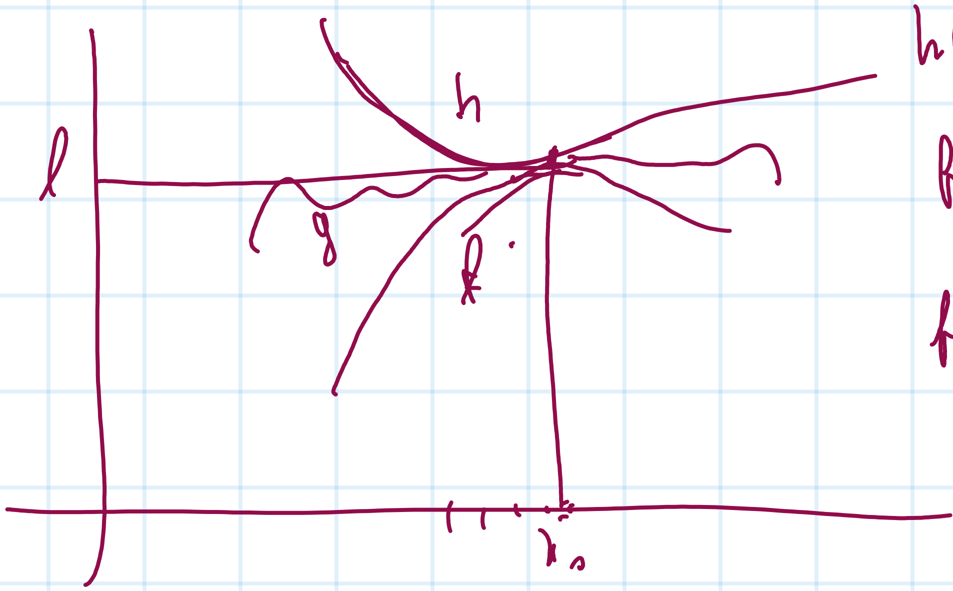
$f(a_n) \to l$ (S)

$g(a_n) \to L$

$\forall (a_n)$ IN $A - \{x_0\}$, $a_n \to x_0$

? $\Downarrow$

$f(a_n) \cdot g(a_n) \to l \cdot L$



$g(a_n) \to \ell$ (?)

$\forall a_n \to x_0$

$h(a_n) \to \ell$

$f(a_n) \to \ell$

$f(a_n) \leq g(a_n) \leq h(a_n)$

$\downarrow \quad \downarrow \quad \downarrow$

$\ell \quad \ell \quad \ell$