Lezione 12: Massimo Limite e Minimo Limite

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   A) DEFINIZIONI

PROSSIMA VOLTA
   B) CARATTERIZZAZIONI EQUIVALENTI
   C) RELAZ. CON ESISTENZA LIMITE
   D) ESEMPI $\sqrt{n}\sin\sqrt{n}$

$$\left(\sqrt{n}-\lfloor\sqrt{n}\rfloor\right)^{\sqrt{n}}$$

$$\sqrt{n}\,\left(\sin\sqrt{n}\right)^2$$

SUCC. DI CAUCHY

DEF DATA (a_n) DIREMO CHE (a_n) È DI CAUCHY (FONDAMENTALE) SE
$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N}$ t.c. $\forall n, m \geq n_0 \quad |a_n - a_m| < \varepsilon$

LEMMA DATA (a_n) SE (a_n) È DI CAUCHY ALLORA È LIMITATA.

DIM PRENDO $\varepsilon = 1$, SICCOME (a_n) È DI CAUCHY

$$\exists n_0 \in \mathbb{N} \text{ t.c. } n, m \geq n_0 \quad |a_n - a_m| < 1$$

IN PARTICOLARE $|a_n - a_{n_0}| < 1$ OVVERO

$$a_{n_0} - 1 < a_n < a_{n_0} + 1$$

$[a_{n_0} - 1, a_{n_0} + 1] = I$

$a_n \in I \quad \forall n \geq n_0$

SIA $M > 0$ t.c.

$$\{a_1, a_2, \ldots, a_{n_0 - 1}\} \subset [-M, M]$$

QUINDI $\{a_n\} \subset [-M, M] \cup I \subset$ INTERVALLO LIMITATO.

TEO. DATA (a_n) È EQUIV. DIRE CHE:

1) (a_n) È DI CAUCHY

2) $a_n \to l \in \mathbb{R}$

$\boxed{\text{DIM}}$ $(2) \Rightarrow \forall \frac{\varepsilon}{2} > 0 \ \exists n_0 \in \mathbb{N}$ t.c. $n \geq n_0 \Rightarrow |a_n - l| < \frac{\varepsilon}{2}$

$(2) \Rightarrow (1)$

$$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \text{ t.c. } n, m \geq n_0 \Rightarrow \underbrace{|a_n - a_m| < \varepsilon}_{(?)}$$

$$|a_n - a_m| = |(a_n - l) + (l - a_m)| \leq$$

$$\leq |a_n - l| + |a_m - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$(1) \Rightarrow (2)$ $(1) \Rightarrow (a_n)$ è LIMITATA $\Rightarrow \exists (a_{n_k})$ t.c. $a_{n_k} \to l \in \mathbb{R}$

(*) $\forall \varepsilon > 0 \ \exists k_0 \in \mathbb{N}$ t.c. $k > k_0$ $|a_{n_k} - l| < \frac{\varepsilon}{2}$

(*) $\exists n_0 \in \mathbb{N}$ t.c. $\forall n, m \geq n_0$ $|a_n - a_m| < \frac{\varepsilon}{2}$

$\bar{n} = \text{MAX}\{n_0, n_{k_0}\}$

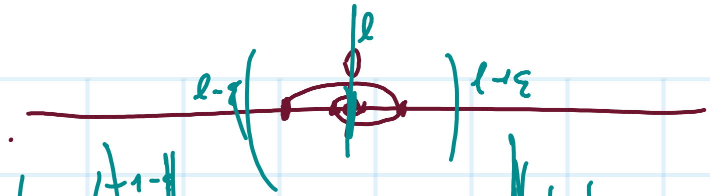
$\forall n \geq \bar{n}$ $\underbrace{|a_n - l| < \varepsilon}$ (?)

PRENDO n_k t.c. $n_k > \bar{n} \geq n_{k_0}$

ALLORA $|a_n - l| = |(a_n - a_{n_k}) + (a_{n_k} - l)| \leq$

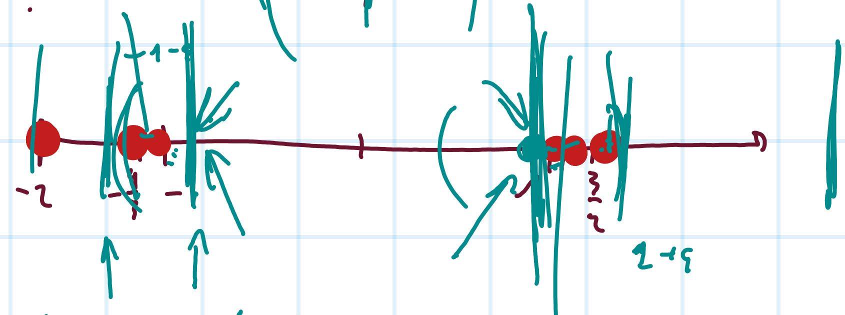
$$\leq \overbrace{|a_n - a_{n_k}|}^{< \frac{\varepsilon}{2}} + \overbrace{|a_{n_k} - l|}^{< \frac{\varepsilon}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$a_n = (-1)^n \cdot \frac{1}{n}$



$a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

$(-1)^n \left(1 - \frac{1}{n}\right)$



DIREMO CHE L È $\limsup_{n \to +\infty}(a_n)$ SE

L_1 $\boxed{D.1}$ - $\forall \varepsilon > 0$ DEF IN n $a_n < L + \varepsilon$ MA FREQ. $a_n > L - \varepsilon$

L_2 $\boxed{D.2}$. SIA $M = \{ \lambda \in \mathbb{R} \mid$ DEF IN n $a_n \leq \lambda \}$ DEFINIAMO

$\limsup_{n \to +\infty}(a_n) = \inf(M)$

$\forall \varepsilon > 0$

$\boxed{D.3}$ SIA $\mathcal{L} = \{ \ell \in \mathbb{R} \mid \exists a_{n_k} \to \ell \}$ DEFINISCO $\exists \ell \in \mathcal{L}$

L_3 $\limsup_{n \to +\infty} \{a_n\} = \boxed{\overset{\text{SUP}}{\text{MAX}} \{\mathcal{L}\}}$ $\{A_n\} \subset M$ t.c. $\ell > L_3 - \varepsilon$

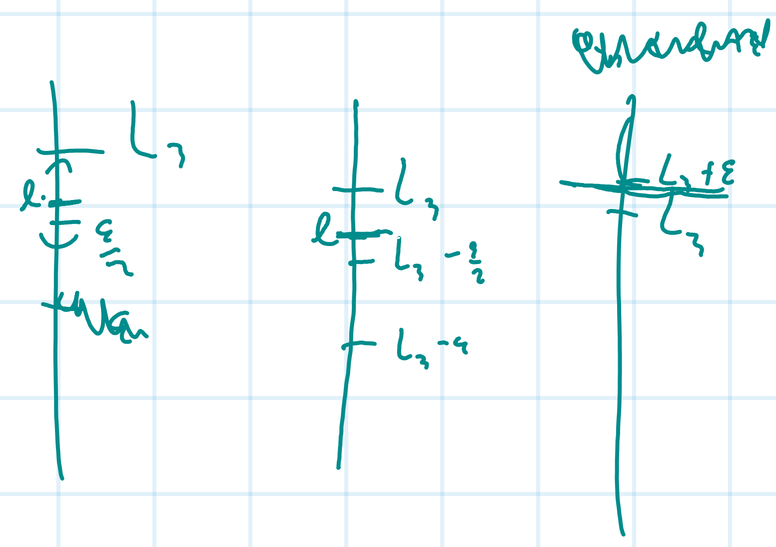
$\boxed{D.4}$ $\limsup_{n \to +\infty}(a_n) = \lim_{n \to +\infty} \left(\underbrace{\sup_{k \geq n} a_k}_{A_n} \right)$ $L_2 \leq L_4$

$\inf \{ \sup_{k \geq n} a_k \mid n \in \mathbb{N} \}$ $L_2 \geq L_4$

$\exists \ell$ t.c. $\boxed{L_3 - \frac{\varepsilon}{2} < \ell}$

FREQ. $a_n > \ell - \frac{\varepsilon}{2} > \boxed{L_3 - \frac{\varepsilon}{2}} - \frac{\varepsilon}{2} =$

$= L_3 - \varepsilon$



M.