Analisi Matematica 1 - 4 Ottobre 2024 - docente: Callegari

Lezione 7: Operazioni tra limiti.

.... DALLA LEZIONE 6

6) TEOREMI DEL CONFRONTO

7) PRIMI ESEMPI DI CALCOLO DI LIMITI COL T. DEL CONFRONTO

A) $\frac{100}{n^2} \to 0$ B) $\frac{\cos(n)}{n+1} \to 0$ C) $n^7 - n^3 \to +\infty$ D) $2^n \to +\infty$ E) $\sqrt{4n^2+3} - 2n \to 0$

INDICE PER LA LEZIONE 7.

1) CASO STANDARD
2) GENERALIZZAZIONE SOMMA ($a_n \to +\infty$)
3) GENERALIZZAZIONI PRODOTTO
   - 3.1 $a_n \to 0$
   - 3.2 $a_n \to +\infty$
4) GENERALIZZAZIONE QUOZIENTE
   - 4.1 $|b_n| \to +\infty$
   - 4.2 $|a_n| \to +\infty$
   - 4.3 $|b_n| \to 0$
   - 4.4 $|a_n| \to 0$
5) FORME INDETERMINATE

ESEMPI: TROVARE LIMITE DI

$A_n = \frac{5n^2 - 2n + 3}{3n^2 - 1}$

$A_n = n + \sin n$

$A_n = \frac{1}{n} \cdot \sin(n)$

$A_n = n \cdot \left(\frac{11}{10} - \sin(n) \right)$

$A_n = \sin(\sin n) / \sqrt{n}$

$A_n = 2^n / (\cos^4 n + \sin^6 n)$

$A_n = \arctan n / \left(\sqrt{n^2 + \sin n} - \sqrt{n^2 + \cos n} \right)$

$A_n = \frac{\sin \frac{1}{n}}{(-1)^n + \sin \frac{1}{n}}$

I° T. DEL CONFRONTO DATE $(a_n), (b_n), (c_n)$ $\ell \in \mathbb{R}$

t.c. 1) DEF. IN n $\underline{a_n \le b_n \le c_n}$

2) $\underbrace{a_n \to \ell, \quad c_n \to \ell}$

ALLORA $\boxed{b_n \to \ell.}$

DIM

$$\left.\begin{aligned} &\forall \varepsilon > 0 \text{ DEF. IN } n \quad \boxed{\ell - \varepsilon < a_n} < \ell + \varepsilon \\ &, \text{ DEF. IN } n \quad \ell - \varepsilon < \boxed{c_n < \ell + \varepsilon} \\ &\text{DEF. IN } n \quad \boxed{a_n \le b_n \le c_n} \end{aligned}\right\} \Rightarrow \text{DEF. IN } n \quad \ell - \varepsilon < a_n \le \boxed{b_n} \le c_n \le \ell + \varepsilon$$

$\uparrow$

$$\forall \varepsilon > 0 \text{ DEF. IN } n \quad \underbrace{\ell - \varepsilon < b_n < \ell + \varepsilon}_{|b_n - \ell| < \varepsilon}$$

II° T. DEL CONFRONTO DATE $(a_n), (b_n)$ T.C.

1) DEF IN n $a_n \le b_n$

→ 2) $\boxed{a_n \to +\infty}$ $(b_n \to -\infty)$

ALLORA $\underline{b_n \to +\infty}$ $(a_n \to -\infty)$

DIM

$$\left.\begin{aligned} &\forall M \in \mathbb{R} \quad \text{DEF. IN } n \quad a_n > M \\ &\text{DEF. IN } n \quad a_n \le b_n \end{aligned}\right\} \Rightarrow \text{DEF. IN } n \quad M < a_n \le b_n$$

$\forall M \in \mathbb{R}$ DEF. IN n $b_n > M$

ES 1 $\quad \frac{100}{n^2} \to 0 \quad$ (SI)

$$\underset{\substack{\downarrow \\ 0}}{0} \leq \frac{100}{n^2} = \underset{\substack{\uparrow \\ \leq 1}}{\frac{100}{n}} \cdot \frac{1}{n} \leq \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}}$$

SE $n \geq 100$

ES. 2 $\quad \frac{\cos n}{n+1} \to 0$

$$-1 \leq \cos n \leq 1$$

$$-\frac{1}{n} \leq \underset{\substack{\downarrow \\ 0}}{-\frac{1}{n+1}} \leq \underset{\substack{\downarrow \\ 0}}{\cos n} \leq \underset{\substack{\downarrow \\ 0}}{\frac{1}{n+1}} < \frac{1}{n}$$

$$a_n \to l$$

$\boxed{\forall \varepsilon > 0 \text{ def. in } n}$

$$l - \varepsilon < a_n < l + \varepsilon$$

$$-l + \varepsilon > -a_n > -l - \varepsilon$$

$$\boxed{-l - \varepsilon < -a_n < -l + \varepsilon}$$

$\boxed{\text{ES. 3}}$

$n^7 - n^3 \to +\infty$

$\bigcirc\!\!\!\!\!\! n \geq 2$

$$\underset{\downarrow \atop +\infty}{\boxed{n^7 - n^3}} = n^3 \underbrace{(n^4 - 1)} \geq n^3 \cdot 1 = n^3 \geq \boxed{n} \to +\infty$$

$n \geq 2 \quad n-1 \geq 1$

$n^4 - 1 \geq 1$

$\boxed{\text{ES. 4}}$

$2^n \to +\infty$

$2^n > n$

$(1+x)^n \geq 1 + nx$

$$\underset{\downarrow \atop +\infty}{2^n} = (1+1)^n \geq 1 + n \cdot 1 > \underset{\downarrow \atop +\infty}{n}$$

ES

$$\underbrace{\sqrt{4n^2+3} - \boxed{2n}}_{a_n} \to 0$$

$$\frac{\sqrt{4n^2+3} - 2n}{1} = \frac{\cancel{4n^2}+3-\cancel{4n^2}}{\sqrt{4n^2+3}+2n} = \frac{3}{\sqrt{4n^2+3}+2n} \leq \frac{3}{\sqrt{4n^2}+2n} = \frac{3}{2n+2n} = \frac{3}{4n} < \frac{1}{n}$$

$$\underset{\downarrow \atop 0}{0} < \underset{\downarrow \atop 0}{a_n} < \underset{\downarrow \atop 0}{\frac{1}{n}}$$

T. OP. SUI LIMITI (STANDARD)

DATE (a_n) (b_n) T.C. $a_n \to \ell$ $b_n \to L$ ALLORA

1) $\underline{a_n + b_n} \to \ell + L$

2) $a_n \cdot b_n \to \ell \cdot L$

3) SE $L \neq 0$ $\frac{a_n}{b_n} \to \frac{\ell}{L}$

$\boxed{\text{DIM}}$

① $\forall \varepsilon > 0$ DEF. IN n $\underline{|a_n + b_n - (\ell + L)| < \varepsilon}$?

$\forall \varepsilon > 0$ DEF. IN n $|a_n - \ell| < \frac{\varepsilon}{2}$

DEF. IN n $|b_n - \ell| < \frac{\varepsilon}{2}$

$$|(a_n + b_n) - (\ell + L)| = |(a_n - \ell) + (b_n - L)| \leq \overbrace{|a_n - \ell|} + \overbrace{|b_n - L|} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

2) $\forall \varepsilon > 0$ DEF. IN n $|a_n \cdot b_n - \ell \cdot L| < \varepsilon$

$$|a_n \cdot b_n - \ell \cdot L| = |a_n b_n - a_n \cdot L + a_n \cdot L - \ell \cdot L| =$$

$$= |\overbrace{a_n (b_n - L)} + \overbrace{L (a_n - \ell)}| \leq$$

$$\leq \overline{|a_n|} \cdot |b_n - L| + \overline{|L| \cdot |a_n - \ell|} \leq$$

$$\leq M \cdot |b_n - L| + \cancel{|L| |a_n - \ell|} \leq \cancel{M} \cdot \frac{\varepsilon}{2\cancel{M}} + |\cancel{L}| \cdot \frac{\varepsilon}{2|\cancel{L}|} = \varepsilon$$

$\forall \varepsilon > 0$ DEF IN n $\quad |b_n - L| < \frac{\varepsilon}{2M}$

$\forall \varepsilon > 0$ DEF IN n $\quad |a_n - \ell| < \frac{\varepsilon}{2|L|}$

$\exists M > 0$ t.c. $\quad \boxed{-M < a_n < M}$

$$\boxed{|a_n| < M}$$

$b_n \neq 0$

3) $\boxed{\frac{a_n}{b_n}} \to \frac{\ell}{L}$ $\qquad \boxed{\frac{1}{b_n} \to \frac{1}{L} \; ?}$

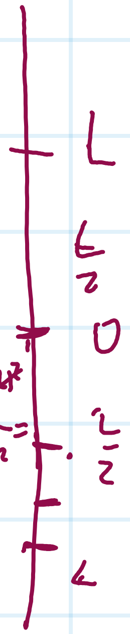
$\forall \varepsilon > 0$ DEF. IN n $\quad \left| \frac{1}{b_n} - \frac{1}{L} \right| < \varepsilon$

$L \neq 0$

$$\boxed{\left| \frac{1}{b_n} - \frac{1}{L} \right|} = \frac{\overline{|b_n - L|}}{\underline{|b_n|} \cdot |L|} \underset{\uparrow}{<} \frac{\overline{|b_n - L|}}{\underline{\frac{1}{2} \cdot |L| \cdot |L|}} < \frac{\varepsilon \cdot \cancel{\frac{1}{2}|L|^2}}{\cancel{\frac{1}{2}|L|^2}} = \varepsilon$$

DEF. IN n

DEF IN n $\quad \boxed{|b_n| > \frac{1}{2}|L|}$



$\forall \varepsilon > 0$ DEF. in n $|b_n - L| < \varepsilon \cdot \frac{1}{2}|L|$

$$\frac{a_n}{b_n} = \underset{\substack{\downarrow \\ \ell}}{a_n} \cdot \underset{\substack{\downarrow \\ \frac{1}{L}}}{\frac{1}{b_n}} \to \frac{\ell}{L}$$

T. (GEN. SOMMA)

DATE (a_n) (b_n) t.c.

1) $a_n \to +\infty$
2) b_n INFERIORMENTE LIMITATA ($\exists \lambda \in \mathbb{R}$ t.c. $b_n > \lambda$)

ALLORA $a_n + b_n \to +\infty$

$\boxed{\text{DIM.}}$

$\forall M \in \mathbb{R}$ DEF IN n $a_n + b_n > M$

DEF. IN n $a_n > M - \lambda$

$a_n + b_n \geq (M - \lambda) + (\lambda) = M$

T. (GEN. PROD ①)

DATE (a_n) (b_n) t.c.

1) $a_n \to 0$

2) b_n LIMITATA $\left(\exists \lambda > 0 \text{ t.c. } |b_n| < \lambda \right.$

ALLORA $a_n \cdot b_n \to 0$

$\boxed{\text{DIM}}$ $\forall \varepsilon > 0$ DEF. IN n $|a_n \cdot b_n - 0| < \varepsilon$

$\boxed{|a_n \cdot b_n| < \varepsilon}$?

$$|a_n \cdot b_n| \le \frac{\varepsilon}{\cancel{\lambda}} \cancel{\lambda} = \varepsilon$$

SO CHE

$\forall \varepsilon > 0$ DEF. IN n

$$|a_n| < \frac{\varepsilon}{\lambda}$$

T. (GEN PROD. 2)

DATE (a_n) (b_n) T.C.

1) $a_n \to +\infty$

2) $\exists \lambda > 0$ t.c. DEF IN n $b_n > \lambda$

ALLORA

$$a_n \cdot b_n \to +\infty$$

$\boxed{\text{DIM}}$

$\forall M > 0$ DEF. IN n $a_n \cdot b_n > M$ (?)

$$\overbrace{a_n} \cdot \overbrace{b_n} > \overbrace{\frac{M}{\cancel{\lambda}}} \, \overbrace{\cancel{\lambda}} = M$$

T (CEM. QUOZ. 1)

DATE (a_n) (b_n)

1) $|b_n| \to +\infty$

2) a_n limitata $\big(\exists \lambda > 0$ t.c. $|a_n| < \lambda$

ALLORA $\frac{a_n}{b_n} \to 0$

$\boxed{\text{DIM}}$

$\forall \varepsilon > 0$ DEF. IN n $\left|\frac{a_n}{b_n}\right| < \varepsilon$

$$\left|\frac{a_n}{b_n}\right| = \frac{|a_n|}{|b_n|} \leq \lambda \left(\frac{\varepsilon}{\cancel{\lambda}}\right) \varepsilon$$

$$\boxed{|b_n| > \frac{\lambda}{\varepsilon}}$$