Analisi Matematica 1 - 3 Ottobre 2024 - docente: Callegari

Lezione 6: Successioni in R e loro limiti (definizioni e primi teoremi, monotonia, confronto, primi limiti notevoli)

INDICE

PER CASA: ESERCIZI 21-33 PAG. 65 ESERCIZIARIO

$\boxed{\text{DEF.}}\boxed{1}$ UNA SUCCESSIONE A VALORI IN $\mathbb{R}$ È UNA FUNZIONE

$$a: \mathbb{N} \to \mathbb{R}$$
$$n \mapsto \underbrace{a(n)}_{a_n}$$

$\boxed{\text{DEF. 2}}$ DATA (a_n) E DATO $\ell \in \mathbb{R}$ DIREMO CHE

1. $\boxed{\text{"}a_n \to \ell\text{"}}$ SE

"$\forall \varepsilon > 0, \underbrace{\exists n_0 \in \mathbb{N} \text{ t.c. } \forall n \geq n_0}_{\text{DEF. IN } n}$, SI HA $|a_n - \ell| < \varepsilon$"

2. "$a_n \to +\infty$" SE
$(-\infty)$

"$\forall M \in \mathbb{R}\ \exists n_0 \in \mathbb{N}$ t.c. $\forall n \geq n_0$ SI HA $a_n > M$" $(<)$

3. (a_n) È INDETERMINATA SE NON VALE NÉ (1) NÉ (2).

$\boxed{\text{OSS. 1}}$ DATO UN PREDICATO $P(n)$ DIPENDENTE DA UN PARAMETRO $n \in \mathbb{N}$ DIREMO CHE

"$P(n)$ VALE DEFIN. IN n" SE $\exists n_0 \in \mathbb{N}$ t.c. $\forall n \geq n_0$ $P(n)$ È VERA

"$P(n)$ VALE FREQUENT. IN n" SE $\forall n_0 \in \mathbb{N}$ $\exists n > n_0$ t.c. $P(n)$ È VERA

ESEMPIO NEGARE "$a_n \to \ell$"

$$\neg\left(\forall \varepsilon>0\ \exists n_0 \in \mathbb{N} \text{ t.c. } \forall n \geq n_0 \text{ si ha } |a_n-\ell|<\varepsilon\right)$$

$$\|$$

$$\exists \varepsilon_0>0 \text{ t.c. } \forall n_0 \in \mathbb{N}\ \exists n \geq n_0 \text{ t.c. si ha } \underline{|a_n-\ell| \geq \varepsilon_0}$$

$$\neg\left(\forall \varepsilon>0 \text{ DEF. IN } n \quad |a_n-\ell|<\varepsilon\right)$$

$$\boxed{\exists \varepsilon_0>0 \text{ t.c. } \underline{\text{FREQ IN } n} \quad |a_n-\ell| \geq \varepsilon_0}$$

T. UNICITÀ DEL LIMITE

DATA (a_n) SE LIMITE DI (a_n) ESISTE ALLORA È UNICO.

DIM

P.A. SUPPONGO CHE $\exists^{\text{no}}\ \ell_1, \ell_2 \in \mathbb{R}$ t.c. ($\ell_1 \neq \ell_2$)

$a_n \to \ell_1 \qquad a_n \to \ell_2$.

$\forall \varepsilon>0$ DEF IN $n \qquad \ell_1-\varepsilon < a_n < \ell_1+\varepsilon$

$\forall \varepsilon>0$ DEF. IN $n \qquad \ell_2-\varepsilon < a_n < \ell_2+\varepsilon$

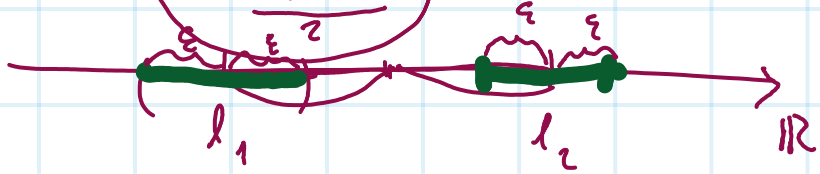
$|a_n-\ell_1|<\varepsilon$

$\Updownarrow$

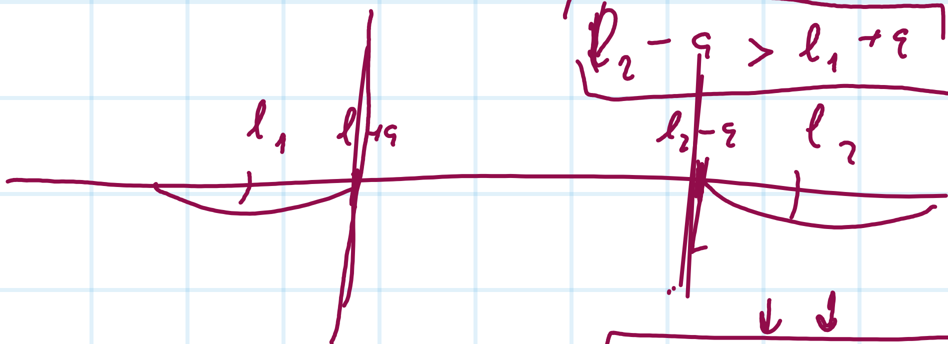
$-\varepsilon < a_n-\ell_1 < \varepsilon$

$\ell_1-\varepsilon < a_n < \ell_1+\varepsilon$

$\varepsilon < \dfrac{\ell_2-\ell_1}{2}$



DEF. IN n $\quad a_n < l_1+\varepsilon \quad a_n > l_2-\varepsilon$



$\boxed{l_2-\varepsilon > l_1+\varepsilon}$?

$\boxed{l_2 - l_1 > 2\varepsilon}$?

$$\boxed{\frac{l_2-l_1}{2} > \varepsilon}$$

T. PERM. DEL SEGNO DATA (a_n) E $l>0$, SE $a_n \to l$ ALLORA DEF. IN n $a_n > 0$

DIM. SIA $\varepsilon > 0$ t.c. $\varepsilon < l$

SO CHE DEF IN n $|a_n - l| < \varepsilon$

$$0 < l-\varepsilon < a_n < l+\varepsilon$$

(PER CASA CASO $l<0$)

T. DI LIMITATEZZA SE $a_n \to l \in \mathbb{R}$ ALLORA $\exists M>0$ t.c. $\forall n \in \mathbb{N}$ $\quad -M < a_n < M$

DIM.

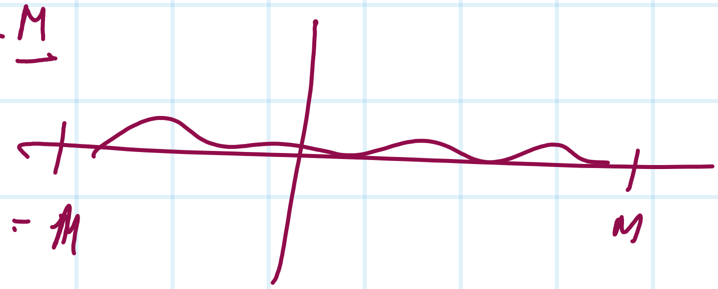
SIA n_0 t.c. SE $n \geq n_0$ ALLORA $l-1 < a_n < l+1$

$$\{a_1, a_2, a_3, \dots, a_{n_0-1}\}$$

$$M = 1 + \max\{|a_1|, |a_2|, \dots, |a_{n_0-1}|, |l-1|, |l+1|\}$$

$$|a_n| = |a_n - 0| < M$$

$$l - \quad < a_n < l+1$$



$\boxed{\text{DEF.}}$ (a_n) È

CRESCENTE SE $\forall n \in \mathbb{N}$ $\quad a_n \leq a_{n+1}$

(STRETT.) $\quad <$

DECR. $\quad \geq$

(STRETT.) $\quad >$

$\boxed{\text{T.}}$ SE (a_n) È CRESCENTE ALLORA $a_n \to \sup\{a_n \mid n \in \mathbb{N}\}$

$\boxed{\text{DIM}}$ $A = \{a_n \mid n \in \mathbb{N}\}$

2 CASI $\quad$ (2) $\sup(A) = +\infty$ $\quad$ (1) $\sup(A)$ FINITO

(1) SIA $\lambda = \sup A$

$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \text{ t.c. } \boxed{\lambda - \varepsilon} < a_{n_0} < \lambda$

$\forall n \geq n_0 \quad a_n > a_{n_0} > \lambda - \varepsilon$

$a_n < \lambda$

QUINDI $\quad \forall n \geq n_0 \quad \lambda - \varepsilon < a_n < \lambda$

$\lambda - \varepsilon < a_n < \lambda + \varepsilon$

$|a_n - \lambda| < \varepsilon.$

$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \text{ t.c. } \forall n \geq n_0 \quad |a_n - \lambda| < \varepsilon$

2 $\quad \sup(A) = +\infty$

$\uparrow$

SIGNIFICA CHE A NON HA MAGGIORANTI

$\forall M > 0 \quad \exists n_0 \in \mathbb{N} \text{ t.c. } a_{n_0} > M$

POICHÈ (a_n) È CRESCENTE SE $n > n_0$ ALLORA

$M \leq a_{n_0} \leq a_n$

$\forall M > 0 \quad \exists n_0 \in \mathbb{N} \text{ t.c. } \forall n \geq n_0 \quad a_n > M$

$\boxed{\text{ESEMPIO}}$ 1 $\qquad a_n = \frac{1}{n}$ $\qquad \boxed{\frac{1}{n} \to 0 \ ?}$

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad t.c. \quad \forall n \geq n_0 \quad \left|\frac{1}{n} - 0\right| < \varepsilon$$

$$0 < \frac{1}{n} < \boxed{\frac{1}{n_0}} < 0$$

$$\boxed{-\varepsilon < \frac{1}{n} < \varepsilon}$$

$$0 < \frac{1}{n_0} < \varepsilon$$

$$\boxed{n > n_0} \qquad \boxed{0 < \frac{1}{n} < \varepsilon}$$

$\boxed{\text{ES. 2}}$

$$\frac{8n^4 + 3n + 2}{2n^4 + n + 1} \to 4$$

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad t.c. \quad \forall n \geq n_0 \quad \left| \frac{8n^4 + 3n + 2}{2n^4 + n + 1} - 4 \right| < \varepsilon$$

$$\left| \frac{\cancel{8n^4} + 3n + 2 - \cancel{8n^4} - 4n - 4}{2n^4 + n + 1} \right| < \varepsilon$$

$$\frac{+n + 2}{2n^4 + n + 1} < \varepsilon$$

$$\boxed{\frac{2n^4+n+1}{n+2} > \frac{1}{\varepsilon}}$$

$n \geq 1$

$$\boxed{\frac{2n^4+n+1}{4n} > \frac{1}{\varepsilon}}$$

$$\frac{2n^4+n+1}{n+2} > \frac{2n^4+n+1}{4n} > \frac{2n^4}{4n} > \frac{1}{\varepsilon}$$

$$\frac{n^3}{2} > \frac{1}{\varepsilon}$$

$$\boxed{n^3 > \frac{2}{\varepsilon}}$$

$$n_0 = \sqrt[3]{\frac{2}{\varepsilon}}$$