Analisi Matematica 1 - 27 Settembre 2024 - docente: Callegari

Lezione 4: Funzione Esponenziale

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FUNZIONE ESPONENZIALE

1) $x \in \mathbb{N}$ $\quad a^n = \underbrace{a \cdot a \cdot \cdots a}$ (n VOLTE)

$x = n$

$$\boxed{a^n = \begin{cases} 1 & \text{SE } n = 0 \\ a^{n-1} \cdot a & \text{SE } n \geq 1 \end{cases}}$$

$$a^n \cdot a^m = a^{n+m}$$

$$a^n : a^m = a^{n-m} \qquad n \geq m$$

$$(a^n)^m = a^{n \cdot m}$$

$$(a \cdot b)^n = a^n \cdot b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$\underbrace{(a \cdot a \cdots a)}_{n} \cdot \underbrace{(a \cdots a)}_{m} = a^{n+m}$$

2) $x \in \mathbb{Z}$ con $n < 0$ $\quad \boxed{x = -n}$

$$a^n = \frac{1}{a^n}$$

$\boxed{a>0}$

3) $x \in \mathbb{Q}$ $\quad x = \frac{m}{n}$ $\quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$

BEN DEFINITA PERCHÉ $\quad \sqrt[n]{a^m} = \sqrt[n \cdot k]{a^{m \cdot k}} \quad \forall k \in \mathbb{N}$

$$a^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{a^m}} = \sqrt[n]{\left(\frac{1}{a}\right)^m}$$

$$\boxed{a^{\frac{m}{n}} \cdot a^{\frac{p}{q}}} = a^{\frac{m}{n}+\frac{p}{q}} \ ?$$

$$\parallel$$

$$\sqrt[n]{a^m} \cdot \sqrt[q]{a^p}$$

$$\parallel$$

$$\sqrt[nq]{a^{mq}} \cdot \sqrt[n \cdot q]{a^{pn}} = \sqrt[n \cdot q]{a^{mq} \cdot a^{pn}} =$$

$$= \sqrt[nq]{a^{mq+np}} = a^{\frac{mq+np}{nq}} = \boxed{a^{\frac{m}{n}+\frac{p}{q}}}$$

PROPR. DI a^q CON $q \in \mathbb{Q}$ E $a > 0$

$\boxed{a > 1}$

1) $a > 1$, $q_1, q_2 \in \mathbb{Q}$, $q_1 < q_2 \Rightarrow a^{q_1} < a^{q_2}$ ($q_1 = \frac{n}{m}$, $q_2 = \frac{p}{q}$)

$$\boxed{a^{\frac{m}{n}} \overset{?}{<} a^{\frac{p}{q}}}$$

$$\Vert$$

$$\sqrt[n]{a^m} \overset{?}{<} \sqrt[q]{a^p}$$

$$\sqrt[nq]{a^{mq}} \overset{?}{<} \sqrt[nq]{a^{pn}}$$

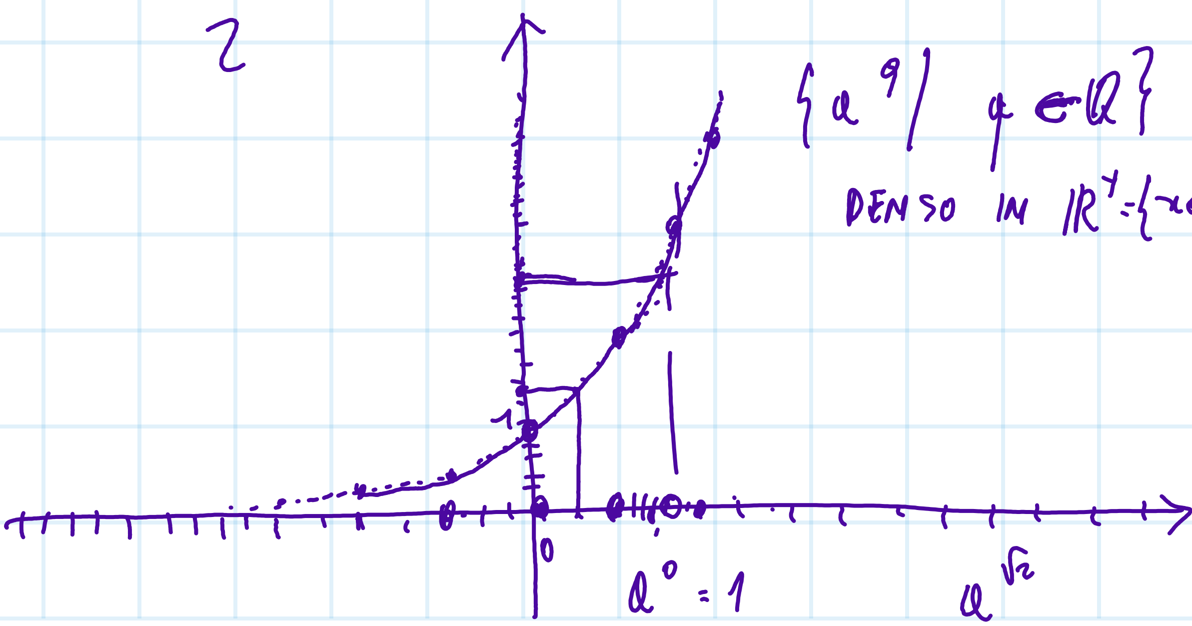
$$a^{mq} \overset{?}{<} a^{pn}$$

$$\frac{1}{q} \cdot \frac{1}{n} \boxed{(mq) \overset{?}{<} (np)} \frac{1}{q} \cdot \frac{1}{n}$$

$$\Uparrow$$

$$\boxed{\frac{m}{n} < \frac{p}{q}}$$

2



$\{a^q \mid q \in \mathbb{Q}\}$

DENSO IN $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$

2) $\{a^q \mid q \in \mathbb{Q}\}$ È DENSO IN $\mathbb{R}^+$ $\quad |a^{\frac{1}{n}} - a^{q}|$

$\boxed{\text{STEP 1}}$ $\forall \varepsilon > 0 \quad \exists n \in \mathbb{N}$ t.c. $\boxed{0 < a^{\frac{1}{n}} - 1 < \varepsilon}$

$1 < \underline{a^{\frac{1}{n}}} < 1+\varepsilon \; ?$

$\forall \varepsilon > 0 \quad \exists n \in \mathbb{N} \quad \underbrace{\sqrt[n]{a} < 1+\varepsilon}$?

$\boxed{a < (1+\varepsilon)^n}$??

DIS. DI BERNOULLI

$x > -1$ $\qquad \boxed{(1+x)^n \geq 1 + nx}$

$\boxed{\text{DIM}}$ SIA $A = \{n \in \mathbb{N} \mid \square \text{ VALE}\}$

1) $1 \in A$ $\qquad 1 + x = 1 + x$

2) $\underbrace{n \in A} \Rightarrow \overset{(?)}{\underbrace{n+1 \in A}}$

$\boxed{(1+x)^{n+1}} = \boxed{\underline{(1+x)}^n} \cdot \underline{(1+x)}^1 \geq$

$\geq \boxed{\underline{(1+nx)}} \, (1+x)^1 =$

$$= 1 + nx + x + nx^2 = 1 + (n+1)x + x^2 > \boxed{1+(n+1)x}$$

$\forall \varepsilon > 0 \ \exists n \in \mathbb{N}$ t.q.

$\exists n \in \mathbb{N}$ t.q.

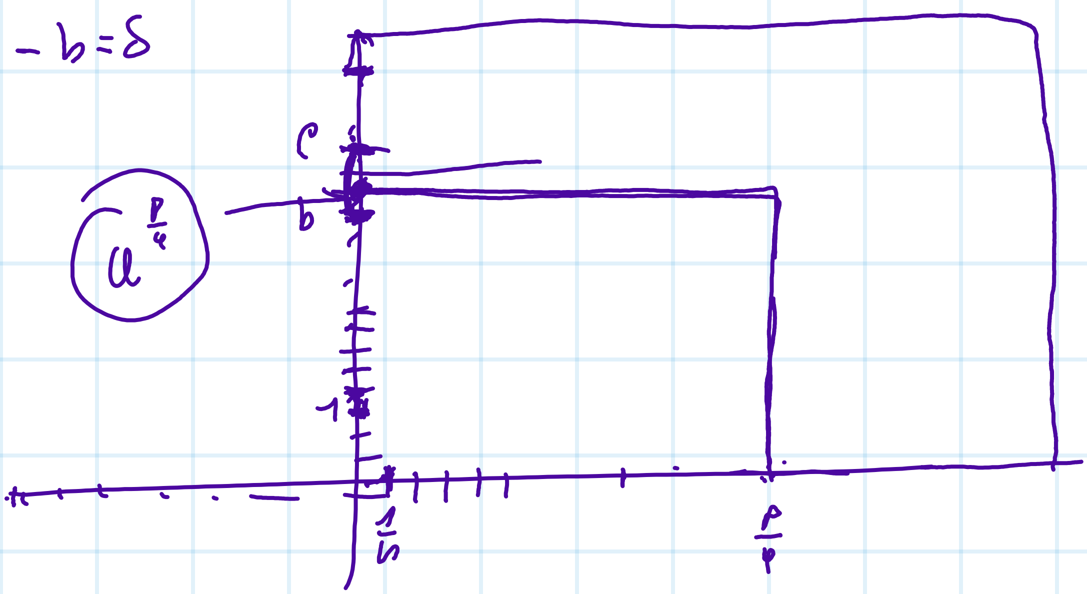
$$(1+\varepsilon)^n \geq 1 + n\varepsilon > a$$

$$n\varepsilon > a - 1$$

$\exists n \in \mathbb{N}$ t.q.

$$\boxed{n > \frac{a-1}{\varepsilon}}$$

$c - b = \delta$



OSS. IMM. DI $\mathbb{Q}$ NON È SUP. LIMITATA

IMM. DI $\mathbb{N}$ NON È SUP. LIMITATA

$$A^n > b$$

$$A^n > c$$

$$a^n = \underline{(1+(a-1))^n} > 1 + n(a-1) > n(a-1)$$

$$a^{\frac{1}{n}} > 1$$

$$\exists k \in \mathbb{N} \quad a^{\frac{k}{n}} > b \quad ?$$

$$\left(a^{\frac{1}{n}}\right)^{k_0} < b < \left(a^{\frac{1}{n}}\right)^{k_0+1}$$

$$\underline{k_0 = \max\{k \in \mathbb{N} \mid a^{\frac{k}{n}} < b\}}$$

$$\left(a^{\frac{1}{n}}\right)^k > b$$

$$\boxed{A} > 1$$

$$a^{\frac{(k_0+1)}{n}} > b$$

$$a^{\frac{k_0+1}{n}} < c \; ?$$

$$\frac{c}{b} > 1$$

$$a^{\frac{k_0}{n} + \frac{1}{n}} < c \quad ?$$

$$a^{\frac{1}{n}} > 1 + \overbrace{\left(\frac{c}{b} - 1\right)}^{\varepsilon > 0}$$

$$\underline{a^{\frac{k_0}{n}} \cdot a^{\frac{1}{n}} < c} \quad ?$$

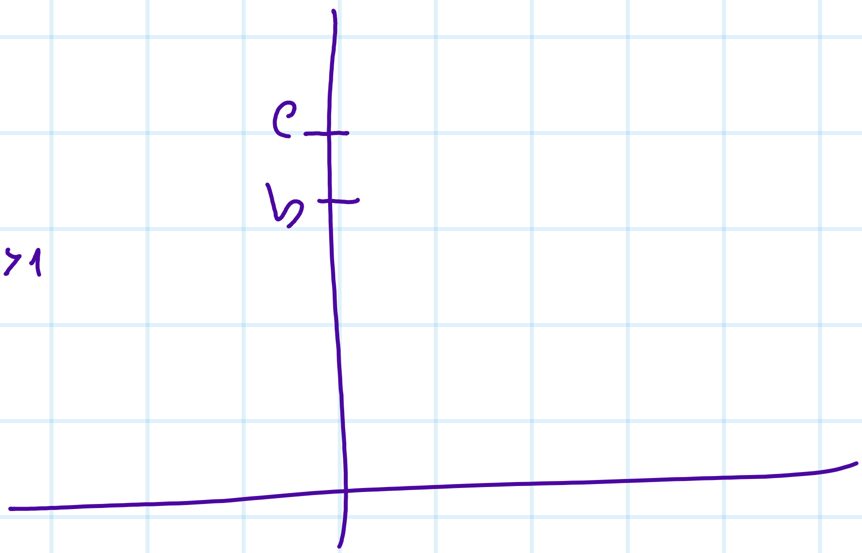
$\exists n \in \mathbb{N}!$ $\boxed{a^{\frac{1}{n}} < \frac{c}{b}}$ $\quad \boxed{b \cdot a^{\frac{1}{n}} < c}$

$$a^{\frac{k_0}{n}} < b$$

$$a^{\frac{1}{n}} < \frac{c}{b}$$

$$a^{\frac{k_0}{n}+\frac{1}{n}} < b\cdot\frac{c}{b} = c$$

$\frac{c}{b} > 1$



$\exists n \in \mathbb{N}$ t.c. $a^{\frac{1}{n}} < \frac{c}{b}$.

k_0 t.c. $\left(a^{\frac{1}{n}}\right)^{k_0} < b < \left(a^{\frac{1}{n}}\right)^{k_0+1}$

$$\left(a^{\frac{1}{n}}\right)^{k} > b$$

$$a^{\frac{k}{n}}$$

$$\boxed{a^{\frac{k_0+1}{n}}} < b\cdot \underbrace{a^{\frac{1}{n}}}_{\frac{b}{c}} < c$$

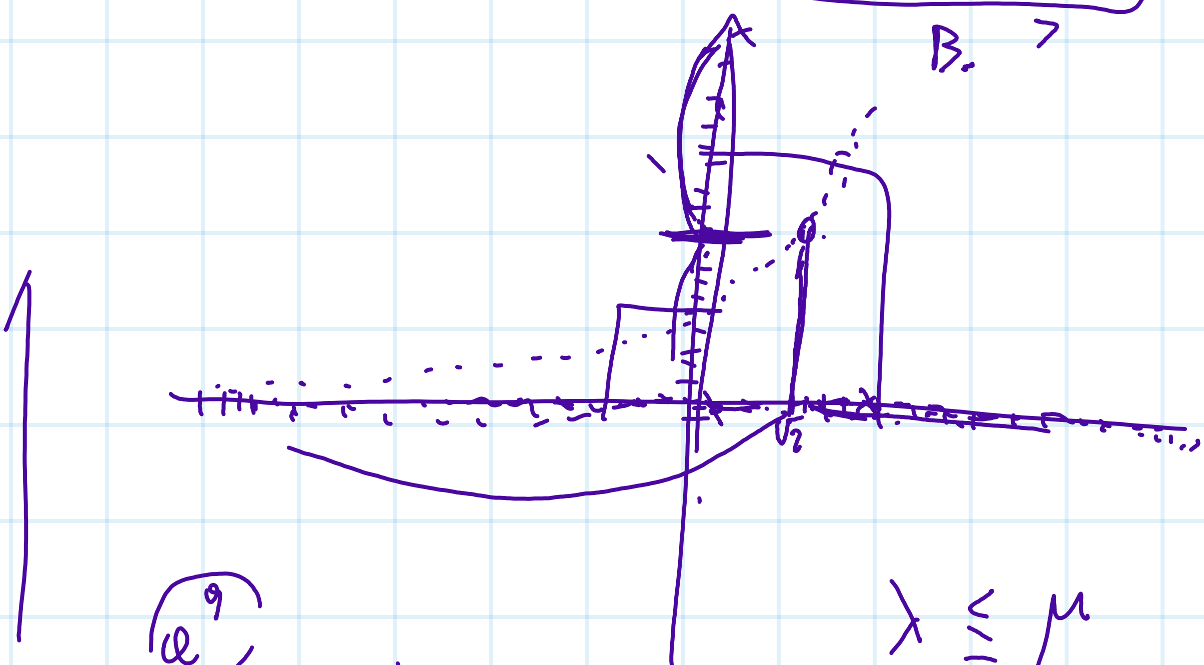
$$A^k = \left(1 + (A-1)\right)^k > (A-1)\cdot k + 1 \geq k(A-1)$$

$$k > \frac{\alpha}{A-1} \qquad k(A-1) > \alpha$$

$$a^{\frac{n_0+1}{n}} = a^{\frac{n_0}{n}} \cdot a^{\frac{1}{n}} < a^{\frac{n_0}{n}} \cdot \frac{c}{b} < \cancel{b} \cdot \frac{c}{\cancel{b}} = c$$

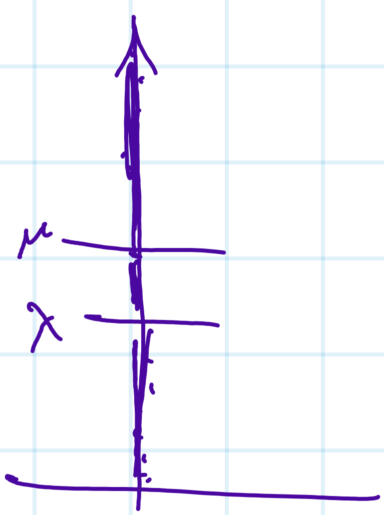
$a^x \qquad x \in \mathbb{R}$

$$a^x = \begin{cases} \boxed{\sup \{ a^q \mid q \in \mathbb{Q},\ q \le x \}} = \lambda & (A) \\ \boxed{\inf \{ a^q \mid q \in \mathbb{Q},\ q \ge x \}} = \mu & (B) \end{cases}$$



a^q

$\lambda \le \mu$



$\lambda \ge \mu$ $\boxed{?}$

$\Leftarrow$

$$a^x \cdot a^y = \boxed{a^{x+y}}$$

$$\lambda \quad \mu \quad \eta$$

$$\le$$

$$\boxed{\lambda \cdot \mu = \eta}$$

$$\boxed{\lambda \cdot \mu \ge \eta}$$

$$\boxed{\lambda \cdot \mu \le \eta}$$

$$\lambda = \sup\{a^p \mid p \le x\} = \inf\{a^p \mid p \ge x\}$$

$$\mu = \sup\{a^q \mid q \le y\} \qquad y$$

$$\eta = \sup\{a^h \mid h \le x+y\} \qquad x+y$$

$$\frac{\varepsilon}{\lambda+\mu} > 0$$

$$\forall \varepsilon > 0 \ \exists p < x \qquad \underline{\lambda - \frac{\varepsilon}{\lambda+\mu} < a^p < \lambda}$$

$$\forall \varepsilon > 0 \ \exists q < x \qquad \mu - \frac{\varepsilon}{\lambda+\mu} < a^q < \mu$$

$$\left(\lambda - \frac{\varepsilon}{\lambda+\mu}\right)\left(\mu - \frac{\varepsilon}{\lambda+\mu}\right) < a^p \cdot a^q = a^{p+q} \le \eta$$

$$\lambda \cdot \mu - \mu \cdot \frac{\varepsilon}{\lambda+\mu} - \lambda \frac{\varepsilon}{\lambda+\mu} + \left(\frac{\varepsilon}{\lambda+\mu}\right)^2 = \qquad a^{x+y}$$

$$= \lambda \cdot \mu - \varepsilon + \frac{\varepsilon^2}{(\lambda+\mu)^2} > \lambda \cdot \mu - \varepsilon$$

$$a^x \cdot a^y$$

$$\varepsilon\left(\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\right) = \varepsilon \frac{\cancel{\lambda+\mu}}{\cancel{\lambda+\mu}} = \varepsilon$$

$\forall \varepsilon > 0$

$\underline{\eta > \lambda \cdot \mu - \varepsilon}$

$\lambda \cdot \mu - \eta < \varepsilon$

$\lambda \cdot \mu - \eta < \varepsilon$

$\underline{\eta} \geq \underline{\lambda \cdot \mu}$

$\boxed{\lambda \mu - \eta \leq 0}$

$\boxed{\eta \geq \lambda \mu}$