Esercitazioni di Analisi 1 - A.A. 21/22 - Callegari

# Lezione 11

9 Dicembre 2021

$o(x^\alpha)\cdot o(x^\beta) = o(x^{\alpha+\beta})$

$x \cdot o(x) = o(x^2)$

$f = o(x) \Rightarrow x\cdot f(x) = o(x^2)$, $\quad \frac{x f(x)}{x^2} \to 0$

$o(x)\cdot o(x) \overset{?}{=} o(x^2)$ $\quad \boxed{x\to 0}$ $\quad f = o(x)$ e $g(x) = o(x) \Rightarrow f(x)\cdot g(x) = o(x^2)$

$o(x^2)\cdot o(x^5) = o(x^7)$ $\quad \frac{f(x)}{x}\to 0 \quad \frac{g(x)}{x}\to 0$ $\quad \frac{f(x)\cdot g(x)}{x^2} = \frac{f(x)}{x}\cdot\frac{g(x)}{x} \to 0$

$\boxed{x\to 0}$

$f(x)$ è $o(x^5) \Rightarrow f(x)$ è $o(x^3)$ $\quad \boxed{\frac{f(x)}{x^5}\to 0}$ $\quad \frac{f(x)}{x^3} = \frac{f(x)}{x^5}\cdot x^2 \to 0$

$e^x = 1 + x + o(x)$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$

$\sin x = x - \frac{x^3}{6} + \dots$

$= x + o(x^\alpha)$ $\quad \boxed{\alpha < 3}$

## O-PICCOLO E O-GRANDE

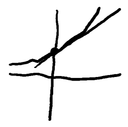
$\boxed{\lim_{x\to 0}\frac{e^x - 1}{x} = 1} \Leftrightarrow e^x - 1 \sim x$ per $x\to 0$

$\Updownarrow$

$\lim_{x\to 0}\frac{e^x - 1}{x} - 1 = 0$

$\Updownarrow$

$\lim_{x\to 0}\frac{e^x - (1+x)}{x} = 0 \Leftrightarrow \boxed{e^x - (1+x) = o(x) \text{ PER } x\to 0}$



$\boxed{y = a + bx}$

$e^x - (a + bx) = o(x)$ $\quad \boxed{x\to 0}$

$a = 1$

$y = 1 + bx$

$\lim_{x\to 0}\frac{e^x - (1+bx)}{x} = \boxed{0} \Rightarrow b = ?$

$\lim_{x\to 0}\left(\frac{e^x - 1}{x} - b\right) = \boxed{1-b} \Rightarrow \boxed{b = 1}$

**DEF** DATA $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$ E DATA $r(x) = a + bx$ DIREMO CHE $y = r(x)$ È RETTA TANGENTE A $f$ IN $x = x_0$ SE

$$f(x) - r(x) = o(x - x_0) \text{ PER } x\to x_0$$

**TEO** DATE $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$ $y = r(x) = a(x-x_0)+b$ È EQUIV. DIRE CHE:

1) $y = r(x)$ È RETTA TANG.

2) $a = f'(x_0)$ E $b = f(x_0)$

$y = f(x_0) + f'(x_0)(x - x_0)$

**DIM** (2) ⇒ (1)

$\lim_{x\to x_0}\frac{f(x) - \overbrace{(f(x_0) + f'(x_0)(x-x_0))}^{r_0(x)}}{x - x_0} =$

$= \lim_{x\to x_0}\left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right) = 0$

(1) ⇒ (2) $\quad \boxed{a(x-x_0)+b}$

$r(x) - f(x) = o(x - x_0)$

$\boxed{r(x) - r_0(x)} =$

$= (r(x) - f(x)) + (f(x) - r_0(x)) = o(x-x_0) + o(x-x_0) =$

$= o(x - x_0)$

$(a(x-x_0)+b) - (f(x_0) + f'(x_0)(x-x_0))$

$\lim_{x\to x_0}\frac{(a - f'(x_0))(x-x_0) - (b - f(x_0))}{x - x_0} = 0$

$\lim_{x\to x_0}(a - f'(x_0)) - \boxed{\frac{b - f(x_0)}{x - x_0}} =$

$\Rightarrow \boxed{b = f(x_0),\ a = f'(x_0)}$

$\boxed{\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}}$

$\lim_{x\to 0}\left(\frac{1-\cos x}{\frac{1}{2}x^2} - 1\right) = 0$

$\lim_{x\to 0}\frac{(1 - \frac{1}{2}x^2) - \cos x}{\frac{1}{2}x^2} = 0$

$\boxed{\lim_{x\to 0}\frac{\cos x - (1 - \frac{1}{2}x^2)}{x^2} = 0}$

$\boxed{y = 1 - \frac{1}{2}x}$

$\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$

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**1** $\boxed{\lim_{x\to 0}\frac{\sin x^2 - \sin^2 x}{x^3}}$ $\quad \boxed{n.64}$ $\quad$ (ANCHE CASO: $\frac{\dots}{x^4\cdot\ln x}$ E $\frac{\dots}{x^4}$)

$= \lim_{x\to 0}\frac{x^2 + o(x^2) - (x + o(x))^2}{x^3} =$

$= \lim_{x\to 0}\frac{\cancel{x^2} + o(x^2) - \cancel{x^2} - (o(x))^2 - 2x\cdot o(x)}{x^3} =$

$= \lim_{x\to 0}\frac{o(x^2) - o(x^2) - o(x^2)}{x^3} =$

$= \lim_{x\to 0}\frac{o(x^2)}{x^3} = ??$

$\lim_{x\to 0}\frac{\sin x}{x} = 1$

$\Updownarrow$

$\to \boxed{\sin x = x + o(x)}$ * $\quad O(x^3)$

$\boxed{g(x)\to 0}$

$\to \boxed{\sin g(x) = g(x) + o(g(x))}$ *

$\lim_{x\to 0}\frac{\sin g(x) - g(x)}{g(x)} = 0$ ?

$\lim_{x\to 0}\left(\frac{\sin g(x)}{g(x)} - 1\right) = 0$

$\sin x = x + o(x)$

$= x + o(x^\alpha)$

$\boxed{\sin x - x = o(|x|^\alpha)}$ $\quad \alpha < 3$ $\quad \boxed{\alpha < 3}$ $\quad \alpha = \frac{5}{2}$

$\sin x = x + o(|x|^\alpha)$ $\quad \alpha < 3$

$\lim_{x\to 0^+}\frac{\sin x^2 - (\sin x)^2}{x^3} = \lim_{x\to 0^+}\frac{x^2 + o(x^{2\alpha}) - (x + o(x^\alpha))^2}{x^3} =$

$= \lim_{x\to 0^+}\frac{\cancel{x^2} + o(x^{2\alpha}) - \cancel{x^2} - (o(x^\alpha))^2 - 2x\,o(x^\alpha)}{x^3} =$ $\quad \frac{f(x)}{x^5}\to 0$, $\quad \frac{f(x)}{x^{3,01}}\cdot x^2$

$\alpha = \frac{5}{2} \to$

$= \lim_{x\to 0^+}\frac{o(x^5) - o(x^5) - o(x^{\frac{7}{2}})}{x^3} =$

$= \lim_{x\to 0^+}\frac{o(x^3) - o(x^3) - o(x^3)}{x^3} = \lim_{x\to 0^+}\frac{o(x^3)}{x^3} = 0$

$0 < \sin x < x < \tan x$ $\quad \boxed{x>0}$ $\quad \alpha < 3$

$0 < \frac{x - \sin x}{x^\alpha} < \boxed{\frac{\tan x - \sin x}{x^\alpha}}$

$\frac{\tan x - \sin x}{x^\alpha} = \boxed{\frac{\tan x - \sin x}{x^3}}\, x^{3-\alpha} \to 0$

**3** $\lim_{x\to 0}\frac{\arctan x^3 - (\arctan x)^3}{x^5}$ $\quad \boxed{n.65}$

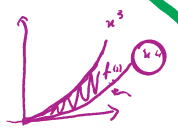
$\alpha < 3$

$\sin x = x + o(x^\alpha)$

$x + \underbrace{O(x^3)}_{h(x)}$

$\left|\frac{h(x)}{x^3}\right| < k$ in intorno di 0

$\frac{h(x)}{x^{2,9}} = \boxed{\frac{h(x)}{x^3}}\cdot x^{0,1} \to 0$



$f(x) = O(x^3)$ per $x\to 0$ $\quad \left|\frac{f(x)}{x^3}\right| <$

$\exists k > 0$, $\exists I$ intorno di 0 t.c. $\left|\frac{f(x)}{x^3}\right| < k$ $\forall x\in[-\delta,\delta]$

**5** $\lim_{x\to 0}\frac{e^x - e^{\sin x}}{x^2}$

**7** $\lim_{x\to 0}\frac{\sin(x + x^3) - \sin x}{x^3}$ $\quad \boxed{n.68}$

**2** $\lim_{x\to 0}\frac{\tan x^4 - (\tan x)^4}{x^5} = ?$

$\boxed{\lim_{x\to 0}\frac{\tan x}{x} = 1}$

$\Updownarrow$

$\boxed{\tan x = x + o(x)}$

$\boxed{0 < \sin x < x < \tan x}$

$\boxed{\alpha < 3}$

$\tan x - x = o(x)$

$= o(x^\alpha)$

$\boxed{x > 0}$

$\boxed{\tan x - x} = o(x^\alpha)$

$0 < \frac{\tan x - x}{x^\alpha} < \frac{\tan x - \sin x}{x^\alpha}$

$\alpha < 3$ $\quad 1 < \alpha < 3$

$\boxed{\tan x = x + o(x^\alpha)}$

$\lim_{x\to 0}\frac{\tan x^4 - (\tan x)^4}{x^5} = \lim_{x\to 0}\frac{x^4 + o(x^{4\alpha}) - (x + o(x^\alpha))^4}{x^5} =$

$= \lim_{x\to 0}\frac{x^4 + o(x^{4\alpha}) - (x^4 + 4x^3\cdot o(x^\alpha) + \dots)}{x^5}$

$= \lim_{x\to 0}\frac{\cancel{x^4} + o(x^{4\alpha}) - \cancel{x^4} - o(x^{\alpha+3})\dots}{x^5} =$ $\quad \alpha = 2$

$= \lim_{x\to 0}\frac{o(x^{4\alpha}) - o(x^{\alpha+3})}{x^5} = \lim_{x\to 0}\frac{o(x^{\alpha+3})}{x^5} = 0$

**4** $\lim_{x\to 0^+}\frac{\sin(\sin x) - x}{x^2\sqrt{x}}$

**6** $\lim_{x\to 0}\frac{\sin(x + x^2) - \sin x}{x^2}$ $\quad \boxed{n.66}$